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Thinking Stories to Support Children's Math Learning

by Herbert P.Ginsburg

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Anna Counts

This is a thinking story about one child's counting from one to 99 (she says she has to ask her mother about what comes after that). A thinking story is a written narrative presented along with embedded and carefully selected video clips–a kind of multimedia case study–that attempts to bring to life the drama of a child's mathematical thinking.

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

A Two Minute Interview with Anna

This is a story with embedded video clips about a girl who is now my favorite counter. At the time of the interview, which took place on an early June near the end of the school year, Anna was a late-four-year-old girl who attended preschool in a low-income neighborhood of New York City. The interviewer was an assistant teacher in her classroom.

The topic of the interview, which lasted less than two minutes, was counting. The goal was to find out how high Anna could count. Why were we interested in this? One reason is that counting is useful. For example, children need to know the counting numbers in order to determine h*ow many*, that is, the number of objects in a collection. If the adult says, "You can have eight cookies," the child can take advantage of the opportunity only if she knows the counting words up to at least *eight* and then can figure how to take no more, and no less, than eight.

A second reason is that the counting words are systematic, embodying important mathematical principles, particularly organizing numbers by tens. Learning the counting words can involve acquiring important mathematical ideas.

It All Begins with One

The interviewer begins with a simple request, namely to count as high as possible.

Video URL: https://player.vimeo.com/video/209857012

The first 10 or so counting words must be memorized in the English language. There is no other way to learn them. Anna knows them all. She begins to use her fingers at *four*, and continues to use them up through *ten*, but then does not use them for eleven and twelve.

Many children use their fingers while counting out loud, although they usually start with *one*. There's nothing wrong with using the fingers while counting, at least up to ten. After all, our counting system owes a lot to the fingers. The origin of our formal base-ten system, that is, a system that relies on groups of ten, is probably biological: we have ten fingers. Indeed sometimes we call numbers *digits*, which is also a word for both fingers and toes. Counting on the fingers becomes cumbersome only after ten. (Toes would be the next practical option, but are hard to count unless you are wearing sandals.) Also, finger counting is valuable because it establishes an early link between the counting words and the things to be counted. The fingers help to establish a *one-to-one correspondence* between each counting word and each thing.

After *ten*, Anna proceeds smoothly and accurately to higher numbers.

Video URL: https://player.vimeo.com/video/209857022

She gets into a kind of wave, a rhythm, for *thirteen* and *fourteen*, and thereafter says the numbers in correct order but seems to run out of steam at *twenty-three*. How exhausted she sounds! But Anna is an indefatigable mathematician and quickly resumes from *twenty-four*.

Video URL: https://player.vimeo.com/video/209857026

Now she rides her wave all the way up to *thirty-nine*. But what's most interesting is her inflection on both *twenty-nine* and *thirty-nine*. She stretches out both words and says them like a question before going on. Clearly she sees those words as special. And they are. Each is the last number in a sequence before the next tens-number must be used. In other words, the counting numbers she is trying to say involve a tens-number, such as *twenty*, followed by the numbers *one* through *nine*. When she elongates the *twenty-nine*, it is as if she is asking, "What's the tens-number I have to say now?" When she gets *thirty*, she tacks onto it the numbers from *one* to *nine*. That's the rule: take your tens-number and just append to it all the unit numbers.

After her question-like "thirty-niiiiiiiiiine," she does something remarkable.

Video URL: https://player.vimeo.com/video/209857036

We might expect her to ask what number comes after *thirty-nine*. Or like other children, she might try out *thirty-ten*, which is not a bad guess: after all, thirty plus ten is forty. Instead, Anna asks what comes after three and answers her own question: "Oh, I know. Four." She does not actually say, *forty*, but continues correctly from *forty-one* to *fifty*.

Why did she ask what comes after three and how did the answer help her? She seemed to know something about the base-ten structure of the numbers. In other words, she seemed to understand that numbers are grouped in tens: the twenties, the thirties, and the forties. She also seemed to know that the first number in each of tens sequences (*twenty*, *thirty*, *forty*) is related to the unit numbers, *two*, *three*, and *four*. In other words, it's as if she thought of the tens-numbers as "two-ten," "three-ten," and "four-ten." If you think of them this way, and you don't know what comes after the "three-ten numbers," you can easily figure out that next must come the "four-ten" numbers. And from there, it's easy to see that "four-ten" is very similar to *forty*. Bingo! It's interesting to note that the Chinese counting system portrays the tens numbers exactly in this fashion: "two-ten, three-ten… nine-ten."

So Anna understood some really important base-ten features of the English counting language. She had uncovered its basic structure. This is one reason for teaching counting out loud, without objects. It's the first case of an abstract mathematical structure that children encounter before school. It has no practical purpose or reward but is interesting in itself. Children greatly enjoy learning to count. There's no reason why they should not pursue counting as a mathematical puzzle. Children are quite capable of this kind of abstract thinking by age four.

Next, Anna was asked to count higher, to sixty. Probably it would have been better not to give her that tens-number, so that we could learn whether she already knows it or could figure it out. In any event, she continued.

Video URL: https://player.vimeo.com/video/209857044

She did her wave and her question-intonation at the end of each tens sequence. She did not know *eighty* but did *ninety* correctly. At the end, she thought that her mom might be able to tell her what comes after *ninety-nine*.

This was quite a performance for a young child. She did many things that are typical for her age group: essentially memorizing the numbers to 20, learning that the unit numbers follow each of the tens-numbers, and pausing before the transition to a new tens-number. Other children make the "thirty-nine, thirty-ten" error and the like. But in my experience, Anna was unique in making the explicit, verbal connection between the unit number and the initial tens-number: *four* and *four-ten*.

So that's the main reason why she is my favorite counter. Another reason, which we tend to overlook, is the sheer joy she had in showing off what she knew and in grappling with the base-ten intellectual puzzle that the counting numbers present.

Finally, watch her entire performance, straight through. The video is just about one minute long but contains a wealth of joyful and meaningful counting. Who knew that this apparently tedious activity could be so interesting?

Video URL: https://player.vimeo.com/video/209857093

Teaching Counting

Here's a method appropriate for those children who can count out loud to at least 20.

Tell the child that they will learn how to count higher than 20, all the way up to one hundred. Count with the child, "twenty-one, twenty-two… twenty-nine." Children will easily learn to add the unit numbers to the tens-numbers, and will rattle off the numbers up to twenty-nine. At this point, do what Anna did: ask the child what comes after two. The child will say, three, and will probably not understand why you are asking. But then you can say that three is like thirty, and then ask what comes after thirty. The child will probably easily get to thirty-nine. After that you can ask what comes after three and point out that four is just like forty (why is forty not spelled "fourty"?). Again the child is likely to produce the numbers after the tens-number very easily.

That's enough for one day. You can repeat as needed and gradually make it up to one hundred. Don't try to explain much to the child. Just point out the similarity. Remember that children may not catch on after only a few attempts. It may take months. Relax. Use the pedagogy of relaxed patience. Believe that the child will eventually learn the ideas and don't worry if she doesn't at first. Be careful not to reward too much. Just acknowledge when the child is right ("That's right, thirty is like three"), and help her to say the right number if she can't get it on her own ("No. After twenty-nine comes thirty. It's just like three.").

Additional Exercises

For additional exercises to accompany this thinking story, visit Using Unedited Videos in Your Courses (/overview/using-unedited-videos-your-courses).

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Ben Learns How to Count

Learning to count is more than memorizing the words from *one* to *ten*. Memory is crucial, but so are ideas. Ben's example can help you understand this.

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

Meet Ben

I interviewed Ben on or very close to his third, fourth, and fifth birthdays. His parents brought him into a room on the Columbia University campus where Ben and I sat at a table to do various "games," as a videographer captured the interactions. His parents stayed in the room, observing and occasionally interacting with Ben. As you will see, the atmosphere did not involve high-stakes pressured testing: Ben clearly enjoyed the interactions, which lasted for quite a long time. (After I interviewed him for about 15 minutes, Janet Eisenband interviewed him for another 30. All this with a boy of three years!) His father told me that after the first session that Ben looked forward to playing again with "Dr. Ginsboo." Here is the story of Ben's counting.

Counting Words

Spoken and written number words permeate the child's everyday world and are used for many different purposes. "Here are two cookies for you" clearly talks about *how many* whereas "Sesame Street is on channel 13" does not. Consider other uses of number such as, "Press the 7 button. We have to go to the top floor." "You can't have all three blocks. Give one to your brother." "Wow, you weigh 36 pounds!" "We're going to read the story of the Three Pigs." "One two, buckle your shoe. Three, four, shut the door." "How high can you count?" "Two, four, six, eight. Who do we appreciate?" Number words have different kinds of meanings that children must figure out.

Early on, children take a great interest in number words and ideas. Some children love counting as high as they can, like grown-ups. They may even be interested in the name of the "biggest" number.

At the age of three, Ben had this unsatisfactory conversation with his mother.

> **Ben:** Mama, but what is the LAST number? The one at the end.
>
> **Christine:** There is no last number. You can count forever without reaching the end.
>
> **Ben:** Mama, are you sure you're listening to me?!

Let's try to listen carefully to Ben. Here is part of an interview with him at age three. Janet asks him to check the number of toy bears. Instead of determining how many, Ben seizes on the opportunity to display his knowledge of counting numbers.

Video URL: https://player.vimeo.com/video/209855266

So at the age of three, Ben rattled off the number words *one* to *ten*, in the correct order. He may have been a bit precocious, but children generally know many number words by this age (Sarneka & Carey, 2008).

A year later, Ben, now four, counts (with his hand in his mouth) up to 17, skipping only 16. After that, with a little help, he makes it up to 20, which he seems to think is the last number.

Video URL: https://player.vimeo.com/video/209855277

Note that I helped Ben by starting to count by myself. I did not assume that he knew nothing when at first he said nothing. Also, after he stopped, I did not assume that he lacked the numbers after 17. I asked him what comes after 17, 18, and 19, and he responded correctly up to 20. This kind of *scaffolding*—gentle hints that help but do not give the answer—can be very useful to learn what a child really knows.

In general, don't assume that children don't know something when they fail to respond, or if they give a wrong answer. They might know much more than you think. Your job is to find ways to dig down beneath the surface to uncover the child's real competence.

We have learned so far that Ben can say the counting numbers up through *twenty*. But what does he know about them? Watch what happens when Ben plays the *catch my mistake* game, first after he has just turned three, and then at age four.

Video URL: https://player.vimeo.com/video/209855923

Video URL: https://player.vimeo.com/video/209855942

At both ages, Ben clearly enjoys turning the tables on an adult, correcting the habitual corrector. The activity is a lot of fun, but also shows that Ben believes several general principles about numbers:

- It is not permitted to skip a number in the sequence. Ben shows that he knows this even at age three when he indicates that something is wrong about saying *twenty* right after *seventeen*, although he probably could not fill in the numbers that are missing.
- It is not permitted to say a number twice. At three, he is particularly worried about how the repeated *three* will multiply, so to speak, in an unmanageable way.

- Numbers are different from colors.

Other children express another important rule: “You have to start with *one*!”

In brief, Ben, along with other children, knows more about counting than is initially evident. He is not only memorizing the numbers but also thinking about what he is memorizing, even at the beginning of his third year. He has figured out, probably by himself, that the category of numbers is different from the category of colors (would anyone have ever explicitly taught him that idea?), and that there are rules for counting, specifically: don’t repeat a number and don’t skip any. And like other children, he is very likely to know the start-from-one rule.

Children’s math, even learning the counting numbers, involves more than memory: independently, young children learn abstract mathematical ideas, even at age three. Parents may be unaware of what their children are learning. Further, as Ben gets older he becomes increasingly able to verbalize his thinking: at age four, he not only recognizes mistakes but also is able to explain *why* they are mistakes. Language and expression are key aspects of math.

At three, Ben is even practicing to be a cognitive psychologist himself when he decides to play the mistake game with Janet as the experimental subject!

Video URL: https://player.vimeo.com/video/209855285

After memorizing the first 12 numbers, learning that number is different from other categories like color, and acquiring rules about skipping and repeating and starting-from-one, English speaking children are faced with a challenge: the words from *thirteen* to *nineteen* are difficult to learn. One approach is to memorize them, which would bring to 19 the total number of number words to be learned in this way. That’s a lot to memorize, but there may be little choice. The alternative approach is to learn the underlying pattern, namely that the unit words (*one* through *nine*) are followed by *teen*, which derives from *ten*. *Thirteen* is a form of “three-ten;” *fourteen* is a form of “four-ten;” and so on until “nine-ten.” But the pattern may be hard to detect and the words are strange. "Thir" stands for three, and *teen* for ten. *Fourteen* is easy, but "fif" in *fifteen* stands for five. It makes no sense to teach the pattern because after 19 the pattern reverses, as we shall soon see. A pox on the “yucky teens!”

At age five, Ben has no difficulty with these strange teen words from *thirteen* to *nineteen*; he rattles off the numbers from one to 29.

Video URL: https://player.vimeo.com/video/209855291

When children are able to count to 20, they have to learn a new rule for generating numbers. The rule is simple, and underlies our base-ten system of number: you first have to begin with the appropriate *tens-word* (*twenty*, *thirty*, *forty,* and so on up to *ninety*), and then you simply append to it the numbers from one to nine. So *sixty-eight* is really “six-ten eight" (which is how it is said in Mandarin, by the way). By contrast, *sixteen* is really “six one-ten,” whereas it would make more sense to be “one-ten six” (again, that’s how it is said in Mandarin). The inventor of the yucky teens rule got it backward! In any event, as they begin to learn the numbers above 20, children begin to grapple with the idea that we can think of numbers as groups of tens and units. The idea also underlies our *place value* system for writing numbers, in which the 3 in 36 refers to three tens and the 6 to six units.

Video URL: https://player.vimeo.com/video/209855296

Ben has trouble with the new rule. Instead of *thirty*, he says *fifty* and goes on from there, correctly, to *fifty-nine*, when he stretches out the *nine*… as he searches for *sixty*. As Ben is trying to figure out what number comes after *fifty-nine*, I pause to let him think. In Ben's case, the silence was not only golden, but also effective.

Video URL: https://player.vimeo.com/video/209855300

After a while, Ben finds *sixty* and goes on to *seventy-nine*, which he says is the last number he can count to. But I'm not so sure. I give him some simple hints.

Video URL: https://player.vimeo.com/video/209855307

At first, I just ask him what comes after *seven*. My intention is to draw a parallel between the unit and tens-numbers. Ben gets it, and then rattles off the numbers to *eighty-nine*. But there he gets stuck and I need to intervene. At first, I try the indirect method of asking him to think about what comes after *eight*, but he cannot make the connection between the *nine* (which of course he knew) and *ninety*. Given this, I then more or less give him the answer and then throw in one hundred as a bonus.

To summarize, learning to count is not simple, at least in English. Children must memorize the first 12 numbers, and along the way learn the rules about repetition and skipping, and the abstract concept that numbers are a special type of word (as opposed for example to the category of colors). Then children have to learn the yucky teens—the very odd numbers from 13 to 19. And next they have to learn the very sensible base-ten pattern from 20 to 99 (although it too has some odd words like *twenty*, which should be "twoty," and *thirty*, which should be "threety"). If children have trouble learning to count, send complaints to the inventors of the English counting words, particularly Professor Yucky Teen.

Teaching Counting

Adults can help children learn counting in several ways. First, children do have to memorize the numbers at least up through twelve. You can use counting books to help them remember, and some simple rhymes, like "One, two, buckle my shoe." You can even help them to memorize counting backwards with "Blast off," as in "Five, four, three, two, one, blastoff!" and later "Ten, nine, eight…. one, blastoff!"

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Ben Learns How Many

The idea of how many is deep, abstract, and cognitively demanding, involving learning number words and figuring out of the number of objects in a collection. Ben illustrates learning what how many really means.

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

Three-year-old Ben could rattle off the numbers from one to 10. He had the numbers memorized, but he did not know how to use them to find out exactly how many toy bears were in front of him. Saying the counting numbers in order is different from accurately counting things. I call the first *counting* (saying the number words, or counting out loud), and the second *enumeration* (finding out *how many*). You can count without being able to enumerate but you cannot enumerate without knowing how to count.

Video URL: https://player.vimeo.com/video/211123618

The Idea of How Many

To an adult, the idea of *how many* seems simple enough. There are four bears here or 10 candies there. But for young children, enumeration is not easy. They must learn methods for figuring out the number of objects in a collection, and they must also learn what *how many* really means. Indeed, the idea of *how many* is deep, abstract and cognitively demanding.

Suppose we have the collection of objects pictured here. We use words to name each. From left to right, we call them *lion*, *school bus*, and perhaps *icky*. If you refer to the school bus as *lion*, people will think that you need some serious help.

Suppose now that we count the objects. We point to the lion and say “one,” to the school bus and say “two,” and to the icky and say “three.” And then we decide to count them again. This time we point to icky first and say “one,” next point to school bus and say “two,” and finally point to lion and say “three.” If we count from right to left, the lion is *three*. And if we count starting in the middle, the lion can

be *two*. The number words are clearly not like names. Names designate specific objects (Harry the lion) or categories of objects (lions), but we should not use number words as names. Neither *one* nor *three* is the name of the lion.

This can be very confusing for young children who are used to words referring to things, and are constantly trying to learn new words. One of my students asked a child to count. She said, “one, two, three, five.” When asked why *four* was missing, the child replied that *she* was four. She seemed to use the counting word as a name for herself.

So one of the first difficulties in enumeration is that counting words are used differently from ordinary names. We can refer to the animal as *lion* but not *tiger* or *football*. But we can refer to it as *one*, *two*, and *three* (or indeed any counting number). This violates ordinary usage and confuses young children (and sometimes adults too, as I will show later).

Why this odd usage? What do the counting words refer to? To understand the issue, consider the basic ideas that underlie counting.

We must say the number words in the proper order. *One* cannot come after *four*. By contrast, the order of counting things does not matter. You can start with icky or you can start with lion.

We have to count each object once and only once. If we use the word *one* to refer to the lion, we can’t use it to refer to the icky as well. I once interviewed a child who, when asked to count some chips, began by arranging them in a circle and then counted the objects over and over again as he looped around the circle (which is the worst way to arrange objects if you want to count them). I had to stop him in order to continue the interview.

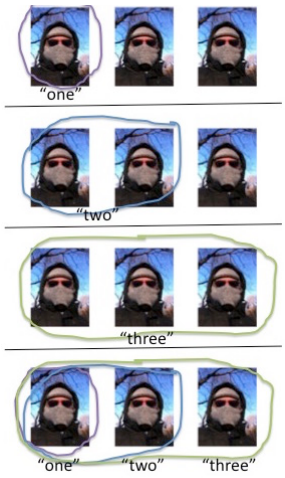
We can count any discrete unit: dogs, imaginary unicorns, or even more imaginary ideas (“I had two ideas today”).

Not only that, we can count any combination of things. A group can include two dogs, 52 unicorns and one idea—or anything else.

Physical arrangement is irrelevant for counting. You can count the objects when they are scattered around or when they are in a line.

The physical nature of the objects does not matter. Every object, no matter what it is, is a single unit for the purpose of counting. This idea must be very strange to little children. Suppose you find that one group has three frogs and the other three horses. Frogs are small and horses are large. How can both groups be *three*?

And finally, here is the really big and difficult idea: each number in the sequence does not refer to an object but to the number of objects (the *cardinal value*) up to that point. Adult counting is so automatic that we sometimes don’t appreciate the complex ideas behind it. But if we are to teach effectively, we must try to see the world from the child’s point of view. In other words, we must overcome our adult egocentrism.



Imagine that you see these pictures of a strange looking arctic creature in an exotic land called *Riverside Park*. You want to find out the number of creatures in total: the basic issue of *how many*.

You begin by drawing a line enclosing the first picture and say "one" because there is one creature. Next you draw a line around the first two and say "two" for the same reason: there are two creatures within. But when you point to the second object and say "two," that object itself is not *two*. The word refers to the numerical value: the cardinal value of the set, the collection of two arctic monsters. Then you draw a line around all creatures and say "three." Again, *three* refers to the whole group, not to the third creature. One way of thinking about this is that you start with a set of one, and then that set of one is included within the larger set of two, and then that set of two (which contains one) is included in the larger set of three.

I'll bet that many readers have not thought about counting things in this way, as sets within sets, always increasing by one. The idea is not simple and it's therefore no surprise that children have trouble with it. Let's see how Ben deals with *how many*.

Ben Enumerates (Figures Out "How Many")

At the outset of the first interview, at age three, Ben displays variable command of enumeration.

Video URL: https://player.vimeo.com/video/211123743

At first, given six "elephants," he counts to five. Not bad: only one off. When the elephants are turned into "bananas," he again counts to five, very quickly, not bothering to count each with care. He spews out the number words very quickly, sometimes, as in the case of the yellow chips, not even looking at what he is pointing to. Yet, he was close, only off by one, on the first two problems: that's pretty good for a child turning three. He is also inconsistent. He gets *three* wrong, calling it *two* but then when I take away one, he says *two* again, and does not appear to be troubled by the inconsistency. And then, at the end of the sequence he succeeds in counting three elephants and four elephants, as he points to one at a time very carefully.

Later, about 25 minutes into the interview (an impressive amount of time for a three-year-old) the interviewer, Janet Eisenband, gives Ben a slightly different enumeration task, asking him to produce a certain number of bears.

Video URL: https://player.vimeo.com/video/211123755

Ben has no problem with producing one and two bears (although he is concerned to select specific bears, perhaps because he has not mastered the rule that a number can refer to any object whatsoever).

But when asked to produce four, he puts out two and then three and finally four (although he referred to the last collection as "four"). Asked to check the last result, his counting goes berserk, as he spews out the number words from one to ten.

Now let's turn to Ben at age four.

Video URL: https://player.vimeo.com/video/211123658

At first, Ben finds it easy to produce the small numbers three and five. Each time, he carefully slides one "banana" towards me and arranges the result in a straight line, carefully setting it apart from the remaining chips. Certainly during the past 12 months, his production has become more deliberate and controlled.

Then we turn to a bigger problem, namely eight chips.

Video URL: https://player.vimeo.com/video/211123625

We see that he gets a little confused after he sets aside six. After all, producing larger numbers requires more working memory (mental space in which to carry out and monitor mental operations on pieces of information) than does production of a smaller number. As children get older, their working memory expands, as does their repertoire of strategies for dealing with the task literally at hand.

When I suggest that Ben check his result, he notices that the chips appear to resemble the numeral 4. That of course does not help him solve the problem, but it shows that he has numerals on his mind. After that, he tries again to put out eight, but this time is a little sloppy about where he puts the chips, failing to push them aside or place them in a straight line so that he would count each chip once and only once. The result is that he gets an incorrect answer. Further, it's not even clear that he is really finished at the end. His inflection indicates uncertainty, and I may have stopped him too soon.

In brief, Ben does well with a relatively large number, but needs to perfect strategies for careful enumeration. One lesson for educators is that children may get wrong answers for many reasons, one of which is that they are sloppy and lack strategies that can guard against carelessness and overcome the limits of working memory. A wrong answer, therefore, doesn't necessarily mean that the child doesn't understand the basic concept of counting, just as a right answer doesn't necessarily mean that the child has mastered counting.

The next important issue is whether Ben understands cardinality, the idea that the last number in the count sequence signifies the total amount. You might expect that, at age three, Ben might not understand it. To find out, I introduce an elephant cardinality problem.

Video URL: https://player.vimeo.com/video/211123698

Ben does not seem to understand cardinality, and each time has to recount the elephants to determine their number, even when it is only two. Also, it's interesting that as he counts the four elephants at the outset, Ben says something like "one, it's a two, it's a three, it's a four." This is very close to saying that each number is the name of that particular elephant with which it is paired.

What about Ben at four years? He was certainly able to enumerate larger numbers than he could at age three. To learn about his understanding of cardinality, I covered up the chips that he had just counted and ask how many are under the sheet of paper.

Video URL: https://player.vimeo.com/video/211123721

After Ben says “zero,” apparently referring to the number of chips on top of the paper, I remind him again that he had previously said that there were eight chips. But he was overwhelmed, and in a gesture of defeat, spread his arms and said, “I give up.”

Next I use Piaget’s classic *conservation of number* problem: will Ben see that the number remains the same even if you simply rearrange the objects into a new and different looking array? This task would seem to be easier than the hiding problem because Ben can now see everything that happens to the objects: nothing is hidden. I asked him to give me five apples and he did.

Video URL: https://player.vimeo.com/video/211123732

In order to determine the number of apples after I rearranged them, Ben had to count them all over again, just as he did when they were hidden. He did not seem to know that the last number counted indicates *how many,* and that moving the objects around does not change that number.

In brief, learning enumeration is complex and difficult. It requires a new use of words to refer to concepts of number (like “fiveness”), not to things. It involves basic and deep mathematical ideas, such as the irrelevance of physical attributes for number: three little things are “more” than two huge things, even though the latter are larger than the former. Children take a long time to master the ideas and the strategies required for enumeration, such as carefully counting each thing once and only once so as to reduce the demands on memory.

Two Lessons Learned

What general lessons can we take from Ben’s story? One is to appreciate that early math is not simple: young children need to grapple with important mathematical ideas and develop strategies for determining *how many*. We saw that at ages three and four, Ben did not fully understand that the number of objects does not change when they are hidden, and that by age four, Ben had developed a strategy to make sure that he counted every object once and only once: he moved objects aside as he counted them.

A second lesson is that a correct answer does not necessarily indicate understanding. Sometimes Ben counted correctly but failed to understand that the last number counted indicates the total amount. A right answer can be a smokescreen for ignorance. As is well known, this phenomenon extends from preschool to graduate school, and well beyond!

How Can We Teach "How Many"?

Children love stories. Stories often have a lot of math embedded in them, as in the case of the three bears. The bears range in size from small (baby bear), to medium (mama bear), and to large (the somewhat overweight papa bear). The bowls of porridge are small, medium, and large. The baby bear gets the small bowl, mama bear gets the medium, and papa bear gets the large and downs it in one gulp. You can say that the bears are one variable (small, medium, large) and that the bowls are a second variable also (small, medium, large). The variables are in a one-to-one relationship in which small bear is paired with small bowl, middle bear with middle bowl, and large bear with large bowl. This is a kind of simple function. Children could not understand the three bears story unless they had some comprehension of relative size (small, medium, large) and simple correlations (one-to-one correspondence, as when small goes with small, medium with medium, and big with big).

You can use stories to teach children *how many*. Click here (https://www.speakaboos.com/story/monster-music-factory) to check out an interactive storybook, the *Monster Music Factory*, which focuses on enumeration.

You can also use planned classroom activities to teach *how many.* Here is one set of organized lessons from *Building Blocks* (Clements & Sarama, 2007):

Students begin by practicing how to make the numeral 4. This in itself is not very interesting, but children do have to learn to write the numerals, and the Building Blocks curriculum wisely introduces written numbers in conjunction with the corresponding number of objects. Thus, the children learn to write "4" in the context of counting four. Notice too that the curriculum introduces counting the different parts of the written numerals (three straight lines, two of which are shorter) and their relative positions (horizontal and vertical). Many things we do and encounter in everyday life, including written numerals and letters, can be “mathematized,” or conceptualized in explicit mathematical terms.

After this, children work with small numbers of objects on plates. They engage in the activity of determining which plate has more objects, and also in determining how many are in each set. The curriculum encourages *subitizing*, that is, seeing the number of the set without counting. Children should be able to glance at a group of objects and know right away that there are two or three or even four. Of course, if children cannot subitize a set, they are helped to count to determine its numerical value.

Finally, pairs children engage in a *get just enough* activity in which they try to find a set that matches the number of another. They might start with four pencils, and then try to find, somewhere else in the classroom, some other group that also has four objects, or they might take four objects (for example, some blocks) from a larger collection.
This activity gives the children a meaningful task to accomplish and encourages them to use counting to solve the problem (that is, to determine whether the new set is the same number or more or less than the four pencils).

Conclusion

How many words are there in this last sentence?

References

There are several research papers that explore what young children know about How Many:

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Huang, Y., Spelke, E. & Snedeker, J. (2010) When is four far more than three. Psychological Science, 17 401--406.

Sarnecka, B. & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. Cognition, 108, 662--674.

Sarneka, B. & Gelman, S. (2004). Six does not just mean a lot: Preschoolers see number words as specific. Cognition, 92, 329--352.

Additional Exercises

For additional exercises to accompany this thinking story, visit Using Unedited Videos in Your Courses (/overview/using-unedited-videos-your-courses).

This website is a project of the **Development and Research in Early Math Education (DREME) Network (https://dreme.stanford.edu/)**.

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Ben Learns How to Add

Addition can grow directly out of children's experience with counting objects. Ben illustrates how a young child adds.

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

How do children learn to add? One kind of addition grows directly out of enumeration. You can think of enumeration—counting the number of things—as adding by ones. You start first with one and add one to it to get two and then add one more to that to get three and so on. When children figure out how many, they at first count one by one. They count all the members of the first group of things, and then continue to count the remaining things in order to determine the sum. But after *counting all* for a period of time, the children then sometimes take shortcuts. Suppose a child sees a bunch of things in front of her. She notices that there are three and then two more on the other side of the table. Her solution is to *count on* from three: “Three… four, five.” So in determining “how many?” children independently discover addition by *counting all* and then *counting on*.

Children can work with several different kinds of addition. In this first example, I ask Ben to determine the number of objects in two different groups of chips. He got the wrong number just a moment before, so I remind him to count very carefully.

Video URL: https://player.vimeo.com/video/224735137

He gets the right answer, five, for the chips in front of him, and then, without counting, seems to “see” that there are three chips in the other pile. When I ask Ben how many there are altogether, he announces that he has to count them, which he does after pushing the piles together, starting with one and getting the right answer. At the outset then, adding is simply the action of combining two groups (making a union of sets) and most children begin by *counting all* (starting with one) to get the sum.

Next, Ben gets extremely excited when he hears that we are going to play a new game. He doesn’t tolerate too much chitchat before he asks, “So what do we have to do?” I hide nine pirate coins, one at a time, in the pirate’s secret bag. Then I tell him that I am adding two more. His job is to figure out how many there are altogether.

[This game is modeled on a wonderful task that was created by Zur & Gelman (2004), who in turn based it on a teacher developed activity.]

Video URL: https://player.vimeo.com/video/224735142

At first, after whispering the numbers from one to nine, Ben again says, “I give up.” But then just a little scaffolding helps him to get the answer. He first asks what is the number after nine, and when given ten, counts on by one more to get the correct answer. Though at first he proclaimed ignorance, he knows enough to ask for the number after nine. And right after that, like many other children his age, he *counted on* to get the next and final number. This means that he has a basic concept of addition as *counting on* from the first addend—at least when he gets a little help from an adult. But didn’t he *count all* in the 5 + 3 addition-as-putting-together task described above? That’s true, but here the task itself, in which both groups of chips are hidden in the bag, channels his thinking into the idea of *counting on*. In other words, because he could not see the chips, and would find it extremely difficult to count their images in his mind, he had almost no choice except to *count on*. So tasks matter: they support or discourage different strategies.

Here’s a slightly different example of Ben’s understanding of addition. This time, I place three chips under a piece of paper, and then two more. He can’t see the result. I ask him to put the same number of chips under his paper. This time, he doesn’t have to tell me the answer; he just has to produce the right number.

Video URL: https://player.vimeo.com/video/224734910

The numbers in this task are much smaller than those in the previous pirate secret bag problem. What is remarkable here is not that he easily gets the sum, but that he is aware of how to solve the problem and can describe the needed method in advance. He says, “We have to go three and then two more.” And he proclaims with some joy that the two sets have the “same number.”

Ben and I continued to play various “games” for about 50 minutes. We were getting ready to wrap up when Ben decided it would be fun (he literally jumped up in his seat) to play another game with Janet Eisenband.

Video URL: https://player.vimeo.com/video/224734922

She shows Ben another pirate money task, a very difficult one. She chooses eight coins from a large pile of pirate booty. She first gives both of them the same small number of pirate coins, namely, four. Then she hides her four and asks Ben to watch as she takes one of his coins and places it with hers. So now he has three and she has five. I thought she was going to ask how many coins are in Janet’s hiding place. But she posed a much more difficult problem that involved equalizing.

She asked him to select from the large pile enough coins to make his collection have the same number as hers. If Janet’s and Ben’s coins are initially the same number, and if Janet takes away one from Ben and adds that one to her collection, then how many coins does Ben need to restore the initial equivalence? Ben’s response surprised me.

Video URL: https://player.vimeo.com/video/224734926

He was able to solve two problems of this type, at least with small numbers. He could have solved the problems by adding everything in his head as he went along. In other words, in the first problem he sees that Janet takes one more for herself, so he adds four to one to get five. He also sees that he lost one, subtracts one from four and gets three. Then he subtracts two from five to find that Janet has two more than he does and that’s the amount needed to make them have the same number once again. Another more general way to solve it is this: Janet took away one from me and gave it to herself.

That means that she has one more than she started with and I have one less. So I need to give myself one to make up for the one she took and another one to make up for the extra one that she got. The problem is complex: it requires some adding, some subtracting, some comparison of magnitudes, and some judgment of equivalence.

We cannot tell exactly how Ben solved the problem. At the end, he does not offer an account of his method. But it is clear that solving the problem involves several important ideas and methods, and also requires a great deal of working memory!

Here's one more example, this time from Ben at the very beginning of his fifth year. We first establish that there are six toy bears under a piece of paper. Then, as he watches, I put two more under the paper and ask him how many there are altogether. Recall that I showed him a very similar problem a year before.

Video URL: https://player.vimeo.com/video/224734937

At first, he is quiet, apparently whispering some numbers to himself. Then he pops up and triumphantly says, "eight!" When I asked him how he knew that, Ben says that four and four make eight, and also that he *counted on* two more from six. I suspect he did that first, and then remembered that four and four is eight. And then he volunteers that, "if you make two more it's ten." When I asked him a series of questions involving the addition of two, he got up to 14 and then made a mistake. Despite this, it's clear that at five years, he is more at ease with addition problems than in earlier clips, seems to understand the equivalence between $6 + 2$ and $4 + 4$, and even creates and solves a new addition problem that begins with his previous answer.

In brief, these examples show that in a sense, young children already know a good deal about addition before they get to elementary school. Most can figure out what happens when you add by combining two sets and what happens when you start with a set and add more to it. They can deal with some abstract tasks, like doing the pirate's secret bag task when one set is completely hidden. They can invent and solve problems on their own. They may even be able to describe their own strategies, although many children are less introspective and expressive than Ben, who may be a bit precocious. Finally, the tasks you give a child make a huge difference. Asking the child to combine two visible sets privileges counting one by one, where as the pirate bag task channels thinking into the more efficient *counting on* strategy.

Why is all this important for you to know? One reason is to understand that children can learn a great deal through their own experience, without direct teaching. You need to give them opportunities to explore and to practice, with no or minimal adult assistance, in a rich environment, with blocks and chips and other things to manipulate, that offers opportunities to engage in meaningful mathematical thinking.

A second reason is to appreciate that children may be more competent in mathematical thinking than we ordinarily imagine. To see this competence you need to give children challenging, carefully designed problems of the kind I have described. If the problems are too easy, children won't have to think much about them.

A third reason is to realize that our math activities must encourage children's language, explanation and reflection. In a sense, math education is literacy education. Children need to learn not only how to do such things as label shapes and count out loud but also to talk about mathematical ideas and their own thinking. And they need to see that you are proud of them for doing interesting work.

Reference

Zur, O., & Gelman, R. (2004). Young children can add and subtract by predicting and checking. Early Childhood

Research Quarterly, 19(1), 121-137.

Additional Exercises

For additional exercises to accompany this thinking story, visit Using Unedited Videos in Your Courses (/overview/using-unedited-videos-your-courses).

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Ethan and the Rocks: One-to-One and Fairness

This video of four-year-old Ethan illustrates how young children engage in mathematics in real-life situations as he distributes rocks "fairly" using one-to-one correspondence.

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

One of the basic principles of number is *one-to-one correspondence*. Suppose that you have some cups and saucers. You put each cup in a saucer—that's a one-to-one correspondence between cups and saucers. You are sharing the cups equally among the saucers. If there are no cups left over and no saucers without a cup, then the number of cups is the same as the number of saucers. It doesn't matter if there are three cups and three saucers or three million cups and three million saucers. In both cases, the cups and saucers have the same number. (It's also true that you wouldn't know where to put the three million cups sitting in the three million saucers, but that's not a math problem: you'd need a bigger kitchen.) The serious point is that math is not only about solving problems in the world around us, but is also about ideas.

Does all this sound abstract? Don't worry. Ethan, age four years and three months at the time of the observation, explains what one-to-one correspondence is all about and how it is related to counting and to justice.

One day when Ethan was visiting his grandfather (me) he rushed into the kitchen saying something, I thought, about rocks. Sensing an interesting mathematical event, I immediately grabbed my iPad and videotaped him (quite clumsily from very odd angles, as you will see). His younger sister was fussing in the background, adding to the normal chaos of children in a small space.

I asked Ethan what he did with the rocks.

Video URL: https://player.vimeo.com/video/225428841

He said that he distributed the rocks to everyone. Not only did he say that "Everybody gets one rock," but also he exuded moral certainty, holding up a finger for emphasis. He was very clear that he didn't give different numbers of rocks to different people. No, he was insistent that the principle of one-to-one correspondence (not in so many words) be applied.

Analyzing videos of behavior is cognitive detective work: you examine the evidence (what you can see children doing and saying) to discover underlying thinking. Because I didn't see what Ethan did outside, I had to rely on what he said. This 15-second episode contains many ambiguities. First, I saw enough to know that he did not in fact give rocks to anyone, certainly not to me. So he apparently was talking about a plan, not a reality. Second, what exactly is the plan? If he did intend to use one-to-one correspondence, how did he know how many rocks to get? Perhaps he simply counted

out the correct number of rocks. If so, he must have already known the number of people in the house. Was that really what he knew and did? Did he know the number of people in the house? Did he then count out the exact number of rocks so that each person would have one rock?

In the absence of further evidence, we cannot know. It was possible that he incorrectly counted the people in the house and/or the rocks, or just took a bunch of rocks without counting them, or even that he made up the whole story. Detectives need to know what they don't know, and they must also recognize when the evidence is ambiguous, such that more than one interpretation may be possible. Perhaps the incomplete evidence will make sense later on when more clues will have been obtained. ("Now it makes sense! The butler was the one who put the poison cream cheese on the bagel.")

So consider what happened next.

Video URL: https://player.vimeo.com/video/225429032

Ethan volunteered a short (14-seconds) but complex explanation: "I counted them on the bench out there so I could make sure I have enough rocks for everybody. I did, so I came in and I gave them to everyone." Again, he really didn't give them to everyone. He was really talking about what he *could* do. In any event, Ethan seemed to mean that counting the rocks could solve his problem of giving one and only one rock to everyone. He even stated that his goal was certainty: he wanted to "make sure." And it's true that if the number of people is the same as the number of rocks, then each person *must* receive a rock, and none will be left over.

It's also important to note that Ethan was able to put his thinking in words, of which he has an unending supply: "counting," "on the bench," "out there," and "enough for everybody." It's not easy for little children to be aware of their thinking and even harder to put it into words that others will understand. Mathematics is partly in the individual's mind, but is also partly a social act requiring clear communication with a community of rational thinkers and speakers.

Next I wanted to find out more about his counting method. So I asked Ethan how many rocks there were altogether.

Video URL: https://player.vimeo.com/video/225429088

As he entered the bathroom to wash his hands, he said that he had in fact not counted but only did the one-to-one correspondence. "I only counted—I only said, 'Julie, Papa, Ethan, Dad, Maya, Mom…" He used a rhythmic cadence to list, one by one, the names of all the people in the house. "I didn't do regular counting. I did just…" I started to question him about the last incomplete statement (I was sure that he was doing a one-to-one correspondence) but there was an interruption (a query from the chef about whether scrambled eggs would be suitable).

What's remarkable here is that he made an explicit distinction between the methods of "regular counting" and one-to-one correspondence. Whether he in fact did what he said is not the point: he was thinking about the difference in methods. Children need to learn to engage in this kind of *metacognition*, being aware of and able to talk about their different methods of finding solutions.

Next I asked him why he gave everybody one rock.

Video URL: https://player.vimeo.com/video/225429139

He explained that he didn't want anyone to have two because then somebody would not have a rock. The explanation was difficult and he got a little entangled in double negatives but eventually he said, "I didn't want nobody to have two because then somebody wouldn't have a rock so that's why I only give one to everybody."

I thought that he was not only concerned with one-to-one correspondence, but perhaps with fairness as well. So I said the following.

Video URL: https://player.vimeo.com/video/225429193

I made an interviewing mistake when I started out by saying that the situation he described (two rocks for one person and zero for the other) would be unfair. I should have let Ethan come to that conclusion himself. Perhaps I put words in his mouth and thereby tainted anything he said next. But the certainty and clarity of his response convince me that he was indeed concerned with fairness and could express his concerns very well, even posing a hypothetical situation. "Imagine if I give you two rocks and I had none." Asked whether he would be upset, Ethan agreed and said, "That's not right. That's why I picked one for everybody."

And then having provided his moral judgment, off he went to watch his favorite television show.

Video URL: https://player.vimeo.com/video/225429244

Although the entire episode was only about two minutes long, its content is rich. The main lessons are that in everyday life, a four-year-old understood one-to-one correspondence; he knew the difference between it and explicit enumeration (counting objects); he knew that the two are related to one another; and he could express and justify his thinking in words.

Moreover, he understood all this even though his knowledge of mathematical symbolism was either minimal or nil. This is not at all atypical. Young children do addition without understanding the + or = signs. The same is true of children and adults in non-literate societies. Clearly it is possible to engage in mathematical thinking without *any* knowledge of the written symbols. By contrast, many students in school use the symbols without understanding the ideas. In a sense, and depending on how math is taught, young children may do more real math before they enter school than after.

We also have learned about the connection between the mathematical and the moral. For Ethan, the idea of one-to-one correspondence was imbedded in the social context and provided a tool for moral judgment and action. Maybe the idea of one-to-one correspondence originated historically in an attempt to regulate social action. We will never know. But it is clear that Ethan used the idea to govern his behavior towards others.

Yet several questions remain. Is Ethan typical? Probably he is not. But my purpose was not to show you the norm but to help you to think about the meaning of one-to-one correspondence, the understanding of it, and the role of language and social context in mathematical activity.

Another question refers to the origins of Ethan's understanding. How did he learn the concept? It's really very difficult to tell: the possible influences are many and must interact in complex ways. In everyday life, he may have faced the arguably universal issues of fairness with respect to the distribution of food, as when the parent says, "You have to have the same as Adam. Each of you gets one cookie." I doubt though whether the parents explicitly described the relation of counting to one-to-one correspondence. Maybe he learned the concept in preschool, through his teacher's intentional

instruction. But research shows that children in cultures without schooling understand concepts like these, so that schooling is clearly not *necessary* for children's understanding of some basic concepts. Or perhaps he *saw* examples of one-to-one correspondence in everyday life, like one foot going in one shoe, and abstracted or constructed the concept from his experience. Or perhaps he learned the concept from the *Team Umizoomi* television show, which he enjoyed and in fact was going to watch at the end of the episode described here. It's hard to disentangle all these threads of experience and influence. There are many, many ways in which children encounter, engage, and learn to solve everyday mathematical problems.

Should Ethan be taught about one-to-one correspondence? He did not seem to need help in talking about it, although many children his age do. One useful approach is to help him formalize his ideas. Soon after the events described, he was helped to produce this egg carton representation. This is a very simple but effective way to structure Ethan's thought and make it explicit in a visual representation. Each rock goes in a compartment of the egg carton, and in each compartment is the name of the person with whom the rock stands in a one-to-one relationship. Maya gets one, Daddy gets one, and so on. Although the egg carton provides a physical, visual representation, it is also very abstract. There is clearly a rock in each of the five compartments, but there is no person in each. The people are imagined and each is *represented* by a piece of paper bearing a written name: the egg carton thus teaches a key literacy skill too. The empty compartments suggest that the one-to-one relationship is not limited to these particular elements; more pairs are possible. The use of the egg carton was a good idea that seized on a *teachable moment* to introduce a first step towards formalization of Ethan's everyday ideas.

So Ethan may not be typical and we cannot say with certainty how his concept developed. But it is clear that in everyday life children may engage with important mathematical concepts in homely ways, without written symbols, to solve problems such as fairness. And it is clear that we can help children to elaborate and even formalize these basic ideas.

Here is the Ethan video in its entirety.

Video URL: https://player.vimeo.com/video/226365229

Several weeks later, Ethan's sister Maya, who was about one year and ten months at the time, did the following, as reported by her mother:

Maya and I went to a store to buy new sippy cups. I picked a two-pack, in which two sippy cups were enclosed in a practically impenetrable packaging of plastic and cardboard. Maya wanted to hold the package in the car on the way home, so I let her. She then set herself to the task of getting the packaging open, all the while saying, "Ethan, Maya...Ethan, Maya." She probably said this about 20 times over the course of 15 minutes. When she finally got the package open, she proclaimed "Ta-da!" When we arrived home and I went to get her out of the car, she handed me one sippy cup, saying "Ethan." Then she hugged the other to herself, saying "Maya."

So Maya was in stage one of one-to-one correspondence and generosity. One for me and One for you equals two special ones.

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What is a Pattern?

Patterns can be notoriously difficult to define. This handout provides illustrations of teacher-child interactions that help take apart (and put back together!) patterns to support our understanding of *pattern*.

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

Suppose we ask a child to define a pattern. What do you think would be the response? Watch this colorful four-year-old, Veronica, as she answers to Janet's question, "Do you know what a pattern is?"

Video URL: https://player.vimeo.com/video/239398434

Veronica never attempts to describe the essence of a pattern, instead trying to create a list involving color names. At first, her list expresses a regular sequence, yellow–red–yellow–red. But then orange gets into the mix, yellow–red–orange, and then yellow repeats itself after orange, and eventually, when she is given a second opportunity to describe pattern, blue fights for a place on the table, as does pink and green, after which general chaos ensues. Maybe it was hard for her to remember the pattern she started with, but in any event Veronica never actually said what makes a pattern a pattern. Before laughing too hard at the child's fumbling attempts, try to define a pattern yourself. It is not at all easy. *Pattern*—like *understanding*—is one of those words that is very general and slippery.Veronica's story is not yet over. Beware of premature conclusions concerning young children's behavior! Prepare for an extraordinary surprise.

Video URL: https://player.vimeo.com/video/239398442

As the interviewer puts out three sets of yellow–blue bears, Veronica watches intensely, and then spontaneously describes the sequence accurately. When the interviewer asks, "Can you put what comes here?" Veronica is able to extend the pattern: yellow–blue–yellow–blue.

Earlier, during her color word extravaganza, Veronica was not able to say anything about the essence of pattern. Yet when dealing with concrete objects, she seemed to understand something important, namely that a pattern can be extended, and she was able to implement the extension accurately. Why the dramatic difference between the two episodes? Note that in the second attempt, the interviewer did not use the word *pattern* and instead simply said, "Can you put what comes here?" This suggests that Veronica's problem is the inability to describe or explain a pattern. At the same time, she has a basic understanding of patterns, at least with regard to the idea of extending a repeating pattern. The child has the intuitive concept but not the words or expressive ability to explain it.

Don't let the absence of correct language fool you. Avoid asking the child to begin with an abstract definition. To learn about the child's understanding, engage her in working with objects. Observe carefully, change the wording of questions, follow up with new questions like, "What are you trying to do here?" Use non-verbal methods also, such as asking the

child to continue a pattern without saying anything about it. In brief, use a variety of techniques to probe the child's understanding. Problems involving manipulatives can uncover abstract thinking. And don't let the child's initial failure determine your assessment of his or her competence.

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A Thinking Story about Seriation and Measurement

Children need to understand some basic mathematical ideas in order to measure accurately. First, they need to know what Goldilocks knows (as revealed by my extensive clinical interview with her), namely that Baby Bear is smaller than Momma Bear and Momma Bear is smaller than Poppa bear; and also that Momma is at the same time smaller than Poppa Bear and larger than Baby Bear. Children also need to learn what Goldilocks does not yet know, namely how to measure the bears exactly. How much smaller is Baby Bear than Momma Bear? And what is this smallness anyway? Is Baby Bear shorter, or less voluminous (skinnier), or lighter, and by how much exactly?

by Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

This thinking story describes how children's basic ideas of small, larger, and largest gradually evolve into sound understanding of measurement. We begin with a rather old video, shot many years ago and not up to current production standards. But this cinéma vérité is very revealing.

Lizbeth and the Staircase

The interviewer gave Lizbeth, who was about three years of age, the *seriation* problem, developed by Jean Piaget many years ago, which involves putting a collection of wooden sticks (or "rods") into a simple series, that is, an ordering from smallest to largest (or largest to smallest).

The interviewer begins with, "Watch this." He then places on the table several red "sticks," and tells a story about building a staircase for a man who wants to climb up so as to go higher and higher. Note that, although saying nothing, Lizbeth is paying close attention both to the story and to what the interviewer did with the sticks.

Video URL: https://player.vimeo.com/video/257442968

At the outset, the interviewer takes care to line up the first three sticks so that the bottoms are on the same level (the base) and the tops increase evenly from left to right. Given this, the fictitious man is able to ascend the steps one at a time, in a very even progression, to get to the highest step. The interviewer then asks Lizbeth to put out the next stick, the bottom of which she does not line up properly next to the bottoms of the other sticks. In response, the interviewer says, very explicitly, "Make it just like this," as he lines up the stick on the base, and then asks her to continue. "Can you make the staircase higher and higher?"

Note that the interviewer modeled the construction of the initial three steps of the series, and also explicitly described the need to line up sticks at the same base and demonstrated how to do so. Given all this, will Lizbeth be able to construct the remainder of the series?

Clip 2 shows that she experiences some difficulty.

Video URL: https://player.vimeo.com/video/257442992

She adds sticks of different sizes, and mostly, they do rise higher and higher, although this is accomplished by ignoring the base. Piaget interpreted this as *centration*, which refers to focusing or centering on only one aspect of the situation (the heights of the tops of the sticks, ignoring the bases) because it is too hard for a young child to keep both in mind (in working memory) at the same time. Also, at the end, even her limited (and presumably easier) focus on the one aspect breaks down as she puts the tops of the last two sticks below the top of the highest stick preceding them. (Advice for interviewers: Don't put the objects so far away from the child so that she has to practically climb over the table to get them!)

But don't jump to conclusions. Watch what happens when, after Lizbeth flounders a bit more, the interviewer does some very direct instruction intended to clarify the need to focus on both base *and* height.

Video URL: https://player.vimeo.com/video/257443006

Lizbeth gets very excited, and in fact utters her first words during this interview, something like "I'll find it." She then carefully lines up each stick on the bottom as she puts it out. She clearly learned something from the instruction. But notice something quite remarkable. Each time, she selects the correct stick in the sequence, the one that is just a little higher than the one before it. To accomplish this, she does not have to try out various sticks by placing them on the base. Her method was simple and logical: when reaching out for the next stick, she chooses the one that is shorter than all those remaining and therefore larger than the last stick already in the series.

Next, the interviewer introduces a new problem, namely to insert a new stick into the already completed series. Piaget found this to be a challenging problem for preschool-age children because solving it requires breaking up the overall configuration that took much thought to produce in the first place. Watch how Lizbeth handles this situation.

Video URL: https://player.vimeo.com/video/257443029

Lizbeth does not have a clear strategy at the outset. She puts the stick at the beginning of the series and then at the end of it. Then, even after being reminded to consider the bottoms of the sticks, she breaks up the series here and there to no good effect. After the interviewer wipes her nose (I am unmasked: what interviewer other than a father would wipe an interviewee's nose?) and also suggests that the stick had to go somewhere in the middle, she breaks up the series, going one by one from long to short, and opens a space between adjacent sticks. As she is doing this, she does not look at the stick to be inserted. When she gets to what she thinks is the right place, she inserts the missing stick. How did she know where to put it? Did she use an image of the stick in her head? I'm not sure.

This interview suggests some important lessons about seriation and about interviewing. First, thinking about increases in size, whatever attribute is involved (length, height, weight, volume, and so on), is more complex than it may initially

appear. It requires what we might call “fair comparisons.” The child must learn, for example, that he cannot stand on a chair to be considered taller than his sister (just as he has to make sure that the sticks need to have a common base).

It also requires understanding these increases in size are orderly, getting bigger and bigger, just as in a growing pattern. The child may also understand, as Lizbeth did, how the smallest in one collection may be the largest when placed in another. And finally, the child needs the flexibility of thought required to examine what she has created and to modify it as necessary. That’s a lot to know, even when the child seems only to be creating a staircase with sticks. And you should see the reasoning that goes into the child’s block play and construction. But that’s another topic!

The episode also teaches some assessment lessons. First, don’t assume that a child’s initial failure is conclusive. Modifying the problem, or even doing a bit of direct instruction, may reveal unsuspected competence. Of course, if you have just taught the child something, you have to take care to determine whether the child’s correct response is a mere copying of what you did or whether the direct instruction essentially helped the child to understand what the task and question were all about. (“Oh, I get it; he wants me to make a staircase that is flat on the ground, as if I made the sticks stand up.”) Second, you can learn a lot about a child’s knowledge by having the child work with objects in response to your questions, even if the child says very little. This is not naturalistic observation (observation of everyday behavior), bur rather observation of behavior in a test-like situation that forms the basis of the clinical interview. Here, Lizbeth “answered” the clinical interview questions by arranging sticks in various ways.

Practice just for you: Maya the Elder

Let’s continue with four-year-six-month-old Maya the Elder (my granddaughter) who is beginning to build a staircase.

Video URL: https://player.vimeo.com/video/257443057

That is all I am going to reveal. This video gives you an opportunity to practice what you have learned and reflect on it. Analyze and interpret what Maya the Elder does. Does she understand the role of the base? Can she insert a stick in the right place? Can she fix mistakes? Can she select the smallest of all those remaining in order to find the largest stick of the staircase already constructed? Can she explain what she has done and why? You figure it out.

Maya the Younger

When you see the interesting thinking of a child at the age of about four, you might forget about its origins just a year or so earlier. Here’s 21-month-old Maya the Younger’s first staircase—sort of.

Video URL: https://player.vimeo.com/video/257443765

She didn't do so well, and her only language was "OK!"

Ethan's unappreciated measurement and negativity

At age four-years-four-months, Ethan (Maya's brother) builds a tower with three-dimensional sticks. When you look closely at them, you can see dividing lines suggesting that each stick is separated into what appear to be individual cubes attached to one another.

(In math talk, we might say that the rectangular prism shows square faces adjacent to one another on the four surfaces. This arrangement gives the appearance of cubes connected to one another.)

At the outset of this episode, Ethan has already built a staircase with six sticks.

Video URL: https://player.vimeo.com/video/257443824

I then ask Ethan to find the next stick without counting. He selects the red stick and looks at it very intensely, obviously trying to count. He asks whether it is OK to count and is told that it is not. Then he takes the red stick and taps the squares in sequence, again obviously counting, which he denies!

After doing the interview, my first reaction was that I had failed to get Ethan to engage in the problem I was interested in, namely whether he could use methods other than counting to solve the problem. I wanted him to do the Piaget seriation task. But on reflection, I realized that he was instead telling me that he was interested in precise measurement drawing upon the standard units (the square faces) provided by the sticks. He could have solved the problem more easily by visual comparison, for example just by holding up the red stick next to the orange. Even so, he was interested in exact measurement, even though I was not. (Advice to interviewers: Avoid imposing your perspective on the child!) His refusal to play my game turns out to be a blessing in disguise because now I can show in this paper how non-counting seriation can lead to and be replaced by exact measurement.

Watch next the extent to which Ethan's numerical measurement approach is extremely sophisticated. At the outset of the video, Ethan has constructed a very neat tower and I ask what comes next. He counts the number of cubes, gets nine for the black and eight for the red. He places the eight at the taller end of the staircase. Then I ask him which of the two remaining sticks, black or blue, comes next, after the red. Let's follow his reasoning.

Video URL: https://player.vimeo.com/video/257443848

He chooses the black this time without having to count the squares. The most likely interpretation is that he immediately sees from the way the sticks are lined up (with a common base) that the black is shorter than the blue and therefore should come immediately after the red. (This is the argument that Lizbeth implicitly uses to the effect that the smallest of the sticks remaining will be the largest of the sticks in the staircase.)

But then, when I push for a reason, Ethan engages in an explicit analysis of the lengths and their significance. He begins

by saying that he knows that the blue stick is 10. (How he knew this is not clear but is not important for this analysis.) Then he holds up the black stick and says that it has nine because it has one less than the 10 and that if it had one more it would be 10. Then he uses some remarkable language to describe the block of nine: “It has a less square so it’s nine…” In effect, he is saying that the nine would equal the 10 if not for its missing square. Further, “a less square” might be taken to indicate that he is using a negative number, saying that 9 – (-1) = 10. You might conclude that this is quite a remarkable use of the double negative. And then he added: “Pretty easy.”

So Ethan solved the problems by both non-numerical means and by exact measurement. Not so easy as he suggests! And his explicit description of his thinking is not so easy either and should be encouraged in young children.

Conclusion

We have traced children’s journeys in developing informal ideas and methods for dealing with series’ of objects, and then formal ideas and methods for exact measurement as well. Enthusiastic Maya the Younger is followed by Lizbeth’s struggles with seriation and her eventual, although mostly non-verbal, grasp of some basic seriation ideas. Then Maya the Elder provides verbal elaboration on seriation and also demonstrates many competencies that you are urged to discover in the video of her performance. Finally, Ethan shows how exact measurement emerges (despite the interviewer’s efforts) to provide a happy ending to the Thinking Story.

This website is a project of the **Development and Research in Early Math Education (DREME) Network (https://dreme.stanford.edu/)**.

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Bella and the Cubes: A Measurement Mystery

Although they perceive length at an early age, children have difficulty measuring it. To do so, they need to learn several basic mathematical concepts, like the idea of a constant, conventional unit of measurement.This handout describes challenges children face as they learn to measure lengths using non-conventional units (in this example, connecting cubes), and those faced by interviewers in trying to determine what children understand about measurement.

by Colleen O. Oppenzato (https://dreme.stanford.edu/people/colleen-oppenzato) and Herbert P. Ginsburg (https://dreme.stanford.edu/people/herbert-ginsburg)

In everyday life, young children notice and focus on size and size differences. Terry might say that this building is “super big.” Tom may say he is littler than his dad. Taniesha might roll playdough into a snake, making it longer and longer, and proclaiming that it is the “hugest ever.” However, while perceiving and attending to such differences, children may have difficulty in quantifying and describing them accurately. Children need guidance to understand and communicate about precise measurement.

Many teachers introduce measurement by having children use non-conventional units such as connecting cubes, rather than conventional units such as inches. One reason is that rulers can be challenging for young children to manipulate and read accurately. Another reason is that conventional units are abstract and may not make immediate sense to children. There are no objects called *inches* in young children's worlds. Objects like cubes, on the other hand, are familiar and present in the daily lives of preschool-aged children and can be used to help them learn to measure. However, as you are about to see, use of manipulatives like connecting cubes is complex and does not guarantee successful learning.

Bella’s Length Measurement

Let’s watch Bella, who at age four years, nine months, has never measured with cubes.

Video URL: https://player.vimeo.com/video/257840497

The interviewer begins by saying that she will show Bella how to measure, and she places two connected cubes next to the green rod. (On the video the rod may seem to be black, not green.) The interviewer says, “I want to find out how

many cubes long this green rod is. Is it two cubes long, or do I need another cube?" Bella answers, "You need another cube." The interviewer then says, "Another, to make them the same length."

Consider what is happening in this first brief segment. The interviewer's intention was to have Bella understand that the task is to figure out the green rod's length in terms of the number of cubes. But the interviewer says at the end of the exchange shown above, "Another, to make them the same length."

Given this statement, Bella then focuses on equivalence of lengths. She says that more than two cubes are needed to make the stack of cubes the same length as the green rod. However, when a third cube is added, she says, "That makes it not the same length because it's a little bit the same length but on this side [gesturing to the bump which connects the cubes] it has a different length." While her attempts at precision are interesting, the interviewer tries to get her to ignore the bump, a minor feature that is not important for the major task, namely to focus on the number of cube units. The interviewer says, "So it's not exactly the same length, but it's pretty close, right? It's about three cubes long, do you think?" Bella agrees and even points out that using four cubes would result in a stack that is longer than the rod.

In summary, Bella clearly interpreted the task as determining whether the length of the stick was the same as the length of the stack of cubes. Although the interviewer's intention was to present a measurement task (determining how many cubes were needed to describe the length of the rod), she talked about comparing lengths a great deal. Even though the interviewer thought she was assessing and scaffolding Bella's measurement method, the interviewer's language unintentionally promoted a focus on equivalent lengths. When assessing what a child knows, it is important to pay attention not only to the child's responses, but also to how the interviewer phrases questions.

The interviewer then asked Bella to measure an orange stick.

Video URL: https://player.vimeo.com/video/257840533

Bella carefully places five cubes against the stick and says, "It's not exactly the same. Look." She then says she thinks she needs one more. She explicitly states that the outcome is uncertain: "…[the cube stack] might be bigger; it might be the same size." After adding the sixth cube, she says, "I think it's now the same size."

Then something interesting happens when Bella is asked how many cubes long the stick is. She counts along the orange stick, "One, two, three, four." The interviewer wonders if Bella miscounted the cubes, so she asks, "Can you point and count? How many cubes do you have there?" Bella correctly counts six cubes. But when asked again if the stick is six cubes long, Bella responds by counting along the orange stick. "This is one, two, three." She knows how to count the cubes, but, to the interviewer's surprise, does not seem to connect the cube length to the length of the stick. It isn't clear what units she may be considering, if any, as she attempts to determine the length of the stick. She heard the interviewer's instructions and question and, as children often do, is trying her best to comply. She seems to understand that some sort of units are involved—that she is being asked how many *somethings* long the stick is. But she does not choose to use the cubes to measure the length and indeed may not even understand that the cubes are measurement units she could use to determine the length of the stick.

A little later, Bella is asked to measure the length of a longer, blue stick.

Video URL: https://player.vimeo.com/video/257840653

Bella continues to be very precise as she lines up the cubes. When she is finished, she announces, "I'm gonna count the

cubes," and counts the stack of 10 cubes accurately. When asked, "So, how long is the blue?" Bella counts "one, two, three, four" along the blue stick. This is both intriguing and confusing. Why is she counting the flat sticks differently than the cube stacks? What unit, if any, is she using to determine the length of the flat sticks?

Video URL: https://player.vimeo.com/video/257840680

The interviewer tries to delve into her thinking by asking, "What are you counting when you count one, two, three, four?" However, as is common among young children, Bella does not explain. She repeats the count sequence, and as she does so, segments the stick into four imaginary lengths. When she says “one” she points to the very end of the first imaginary segment. She says “two” at the end of the second segment, and so on. She seems to have a rough idea of a unit and believes that it is relevant for measurement.

When the interviewer asks another clarification question, "But you said this is ten cubes long?" Bella makes a distinction between the stack of cubes and the stick. Gesturing to the former, she says, "This is 10 cubes long" and gesturing to the latter, she says, "And this is four." So, there is a disconnect: despite the interviewer's efforts, Bella does not see how the stack of cubes offers measurement units that can be used to determine the length of the stick.

Eager to find out what units she may be using, the interviewer points to the blue stick and asks, "This is four what?" Bella replies, "The blue is four." When asked, "Four cubes?" she starts to agree, but then corrects herself, saying it is not four cubes. She holds up the blue stick and says, "This is four!" And then referring to the cube stack, she says. "This is ten."

Next, the interviewer decides to ask her the same kind of questions about a smaller stick. Sometimes children display more competence with small number problems than large.

Video URL: https://player.vimeo.com/video/257840724

When asked about the orange stick, Bella again counts in segments, and says it is "one, two, three." She then counts the cubes beside the orange stick and says that the stack is six cubes long. Even though Bella lines up the cubes very accurately, she does not see the cube stack as providing units that can be used to measure the stick’s length. So the orange stick is three mysterious units long, but the cube stack beside it is six. While it is unclear how she is measuring the blue and orange sticks, it is interesting that she says that the longer blue stick is four mysterious units long and the shorter orange stick is three mysterious units long. Her actions and descriptions suggest that she has some notion, however imperfect, that longer means more units and shorter means fewer.

Bella’s disconnect between the cubes and sticks still puzzled the interviewer, who therefore asked her to compare the sticks.

Video URL: https://player.vimeo.com/video/257840748

When asked, "So, which is longer, the blue or the orange?" Bella places all of the sticks and stacks of cubes together and says, "This one," pointing at the stack of 10 cubes, not at the blue stick. She says she knows this because "I measured it and this one [the stack of 10] is taller." She seemed to be comparing overall lengths, which in this case is easy to do without counting. Then, when asked if she could look at the number of cubes to compare them, she treats them as quantities, saying 10 cubes is longer than six cubes because the count sequence "one, two, three, four, five, six, seven, eight, nine, ten" is longer than "one, two, three, four, five, six." So she is able to count and understand quantities, but does

not appear to use them in comparing the lengths.

Bella's thinking about measuring the sticks still remains a mystery. But the interviewer uncovered some of her thinking. She can see the difference between the cube stacks. She can determine the number of cubes in each stack and knows which number is higher by referring to the relative length of the counting word sequence. But she does not use the cubes as units of measurement for the length of the sticks.

Conclusions

Summarizing the disconnect

Bella's behavior is interesting because she was successful in creating and counting the connecting cubes but did not use them as non-standard units to measure the length of the sticks. Bella worked with several different approaches: she could compare the lengths of sticks by looking; she could count the cubes; and she could count her mysterious length units. She seemed to understand that some kind of unit was needed to measure the sticks, but the nature of her unit is a bit of mystery. Further, it is clear that she did not make the connection between the use of cubes as a unit of measure and stick length. So she knows a good deal about length comparison, about counting objects, and about the need for measurement units, but does not appear to integrate what she knows into effective measurement with non-standard units.

What explains Bella's approach? One possibility is that the interviewer biased or confused Bella by inadvertently suggesting, in the first episode, that Bella should focus on length comparisons, and not on using the cubes to measure length. The argument is that if the interviewer had not done this or had used a different task, Bella would have used the blocks as non-standard measurement units to measure accurately throughout the interview.

We think that this argument is implausible. The main reason is that after the first episode, the interviewer made several attempts to make clear the goal of using the cubes to determine the length of the sticks. Perhaps the interviewer could have done this more effectively, but watching all of the clips leads us to conclude that Bella's approach did not simply result from misunderstanding of the instructions, but from failing to appreciate the cubes' use as a measurement tool.

But consider other possibilities. One is that if only different manipulatives were used, the child's competence would be revealed. For example, maybe she would have succeeded if she had been given ordinary non-connecting blocks or small square tiles. Or maybe she would have succeeded if the instructions were clearer. The interviewer should not give up on the child too easily because children often possess competencies of which adults are unaware and which require clever methods of assessment to uncover. Children are often smarter than we think, and we need to get smarter to appreciate their abilities.

At the same time, we must consider the real possibility that Bella actually does not understand the use of non-standard

units for measurement. After all, the video shows Bella's very first attempts at measurement with manipulatives. We have no doubt that, over time, with the help of her teacher, Bella can come to learn that a continuous quantity, like the length of a stick, can be described precisely—that is, measured—by a specific number of arbitrary units.

What children need to learn about measurement

Measurement is more complex than it might initially appear. Children need to appreciate that the stick is not just little or big and it is not just longer or shorter than another stick. It is exactly five units long or exactly three units longer than another stick. Children also need to learn that the arbitrary units, in this case cubes, must be placed end to end, with no overlaps or spaces between them. The cubes must span the entire length of the object being measured. And most importantly, children must understand the idea that the total number of cubes represents the length of the object being measured. After children understand these ideas, they will be prepared to understand conventional measurement by an inch or metric ruler. They will realize that the inch or centimeter on the ruler is just like the cube: it is a unit by which to measure a length.

The use of manipulatives

Many early childhood curricula suggest using common manipulatives such as connecting cubes to introduce measurement. Indeed, Bella demonstrated a great deal of competence with making cube stacks that were equal in length to the sticks. However, despite using the cubes to complete the tasks, Bella did not actually measure with them. As the videos show, it's not enough to work with a manipulative; the manipulative must be used to help the child understand the need for, and role of, artificial units (like cubes or inches) in measuring length. By itself, lining up objects with cubes does not guarantee understanding of measurement. Manipulatives can be a wonderful resource, but be careful to use them in a purposive, conceptual manner.

Your goal should be to use intentional teaching to connect use of the manipulatives to the ideas of measurement. Some children will figure out on their own how to use manipulatives as a measurement tool. Many will need help.

One strategy is to first introduce the cubes and ask children to play with them and perhaps use them to make "something interesting." After they are comfortable with the cubes, present children with the task of measuring the length of books (or some other object). Organize children into small groups and give each group several books of different sizes. Some books will be both longer and wider than others. Let's forget about depth for now and just deal with two dimensions. Ask each group to use the cubes to figure out the exact length and width of the books. Go around the room and help the children to place the cubes correctly along the sides of the books (starting at the origin), to count the cubes carefully, and to write down the number of cubes (plus a little left over, if any) they lined up along each side of the book. In this way, you can help the children understand, in ways Bella may not have, that the line of cubes provides information about the lengths and widths of the books, and that this process is *measurement*.

You can go continue over a period of days. For example, you can have children compare measurements of the different books (how many cubes longer is José's book than Jenny's?), then talk about their measurements and explain their use

of the cubes, then collect data on length and width, represent the data in graphs, read the graphs, and finally extend the measurement activity to other objects in the classroom.

Enjoy!

Additional Exercises

For additional exercises to accompany this thinking story, visit Using Unedited Videos in Your Courses (/overview/using-unedited-videos-your-courses).

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